A NON-FINITELY GENERATED ALGEBRA OF FROBENIUS MAPS

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1. Introduction

The purpose of this paper is to answer a question raised by Gennady Lyubeznik and Karen Smith in [LS]. This question involves the finite generation of a certain non-commutative algebra which we define below (cf. section 3 in [LS].)

Let S be any commutative algebra of prime characteristic p. For any S-module M and all $e \geq 0$ we let $\mathcal{F}^e(M)$ denote the set of all additive functions $\phi : M \to M$ with the property that $\phi(sm) = s^{p^e}\phi(m)$ for all $s \in S$ and $m \in M$. Note that for all $e_1, e_2 \geq 0$, and $\phi_1 \in \mathcal{F}^{e_1}(M)$, $\phi_2 \in \mathcal{F}^{e_2}(M)$ the composition $\phi_2 \circ \phi_1$ is in $\mathcal{F}^{e_1+e_2}(M)$. Note also that each $\mathcal{F}^e(M)$ is a module over $\mathcal{F}^0(M) = \operatorname{Hom}_S(M, M)$ via $\phi_0\phi = \phi_0 \circ \phi$. We now define $\mathcal{F}(M) = \oplus_{e\geq 0}\mathcal{F}^e(M)$ and endow it with the structure of a $\operatorname{Hom}_S(M, M)$-algebra with multiplication given by composition.

In section 2 below we construct an example of an Artinian module over a complete local ring S for which $\mathcal{F}(M)$ is not a finitely generated $\operatorname{Hom}_S(M, M)$-algebra, thus giving a negative answer to the question raised in section 3 of [LS].

2. The example

Let $\mathbb{K}$ be a field of characteristic $p > 0$, $R = \mathbb{K}[\![x, y, z]\!]$, and let $I \subseteq R$ be an ideal. Let E be the injective hull of the residue field of R and let f denote the standard Frobenius map of E (cf. section 4 in [K].) Write $S = R/I$ and let E_S be the injective hull of the residue field of S.

Notice that as S is complete, $\mathcal{F}^0(E_S) = \operatorname{Hom}_S(E_S, E_S) \cong S$; the S-module $\mathcal{F}^e(E_S)$ of p^eth Frobenius maps on E_S is given by $(I^{[p^e]} : I)f^e$ (cf. section 4 in [K].)

For all $e \geq 1$ write $K_e = (I^{[p^e]} : I)$. We define

$$L_e = \sum_{\substack{1\leq \beta_1,\ldots,\beta_s<e \\ \beta_1+\cdots+\beta_s=e}} K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} K_{\beta_3}^{[p^{\beta_1+\beta_2}]} \cdot \cdots \cdot K_{\beta_s}^{[p^{\beta_1+\cdots+\beta_{s-1}}]}.$$

Proposition 2.1. *Fix any $e \geq 1$, and let $\mathcal{F}_{<e}$ be the S-subalgebra of $\mathcal{F}(E_S)$ generated by $\mathcal{F}^0(E_S), \ldots, \mathcal{F}^{e-1}(E_S)$. We have $\mathcal{F}_{<e} \cap \mathcal{F}^e(E_S) = L_e f^e$.*

Proof. Any element in $\mathcal{F}_{<e} \cap \mathcal{F}^e(E_S)$ can be written as a sum of elements of the form $\phi_1 \cdot \cdots \cdot \phi_s$ where for all $1 \leq j \leq s$ we have $\phi_j \in \mathcal{F}^{\beta_j}(E_S)$ $(1 \leq \beta_j < e)$ and $\beta_1 + \cdots + \beta_s = e$.

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Each such ϕ_j equals $a_j f^{\beta_j}$ where $a_j \in K_{\beta_j}$, so

$$\phi_1 \cdot \dots \cdot \phi_s = a_1 f^{\beta_1} a_2 f^{\beta_2} a_3 f^{\beta_3} \cdot \dots \cdot a_s f^{\beta_s} = a_1 a_2^{p_1^{\beta}} a_3^{p^{\beta_1+\beta_2}} \cdot \dots \cdot a_s^{p^{\beta_1+\dots+\beta_{s-1}}} f^{\beta_1+\dots+\beta_s} \in L_e f^e$$

so $\mathcal{F}_{<e} \cap \mathcal{F}^e(E_S) \subseteq L_e f^e$.

On the other hand, for all $1 \leq \beta_1, \dots, \beta_s < e$ such that $\beta_1 + \dots + \beta_s = e$,

$$K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} K_{\beta_3}^{[p^{\beta_1+\beta_2}]} \cdot \dots \cdot K_{\beta_s}^{[p^{\beta_1+\dots+\beta_{s-1}}]} \subseteq (I^{[p^{\beta_1+\dots+\beta_s}]} : I) = (I^{[p^e]} : I)$$

so $L_e f^e \subseteq (I^{[p^e]} : I) f^e = \mathcal{F}^e(E_S)$. A similar argument to the one in the previous paragraph shows that we also have

$$K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} K_{\beta_3}^{[p^{\beta_1+\beta_2}]} \cdot \dots \cdot K_{\beta_s}^{[p^{\beta_1+\dots+\beta_{s-1}}]} f^e \subseteq \mathcal{F}_{<e}$$

and we deduce that $L_e f^e \subseteq \mathcal{F}_{<e} \cap \mathcal{F}^e(E_S)$. □

Fix now I to be the ideal generated by xy and yz. We show that $\mathcal{F}(M)$ is not a finitely generated S-algebra.

Proposition 2.2. *For all $e \geq 1$, K_e is generated by*

$$\left\{ x^{p^e} y^{p^e-1}, x^{p^e-1} y^{p^e-1} z^{p^e-1}, y^{p^e-1} z^{p^e} \right\}.$$

Proof. For any $q > 1$,

$$\begin{aligned}
(x^q y^q, y^q z^q) : (xy, yz) &= ((x^q y^q, y^q z^q) : xy) \cap ((x^q y^q, y^q z^q) : yz) \\
&= (x^{q-1} y^{q-1}, y^{q-1} z^q) \cap (x^q y^{q-1}, y^{q-1} z^{q-1}) \\
&= (x^q y^{q-1}, x^{q-1} y^{q-1} z^{q-1}, x^q y^{q-1} z^q, y^{q-1} z^q) \\
&= (x^q y^{q-1}, x^{q-1} y^{q-1} z^{q-1}, y^{q-1} z^q)
\end{aligned}$$

□

Theorem 2.3. *The S-algebra $\mathcal{F}(E_S)$ is not finitely generated.*

Proof. It is enough to show that for all $e \geq 1$, $\mathcal{F}(E_S)$ is not in $\mathcal{F}_{<e}$ and we establish this by showing that the generator $x^{p^e} y^{p^e-1}$ of K_e is not in L_e.

Since L_e is a sum of monomial ideals, $x^{p^e} y^{p^e-1} \in L_e$ if and only if $x^{p^e} y^{p^e-1}$ is in one of the summands. So we now fix $e \geq 1$ and $1 \leq \beta_1, \dots, \beta_s < e$ such that $\beta_1 + \dots + \beta_s = e$, and show that the ideal

$$K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} K_{\beta_3}^{[p^{\beta_1+\beta_2}]} \cdot \dots \cdot K_{\beta_s}^{[p^{\beta_1+\dots+\beta_{s-1}}]}$$

does not contain $x^{p^e} y^{p^e-1}$.

Since z does not occur in $x^{p^e} y^{p^e-1}$, it is enough to show that with $J_e = x^{p^e} y^{p^e-1} R$,

$$J_{\beta_1} J_{\beta_2}^{[p^{\beta_1}]} J_{\beta_3}^{[p^{\beta_1+\beta_2}]} \cdot \dots \cdot J_{\beta_s}^{[p^{\beta_1+\dots+\beta_{s-1}}]}$$

does not contain $x^{p^e} y^{p^e-1}$. The exponent of x in the generator of the product above is

$$p^{\beta_1+(\beta_1+\beta_2)+\dots+(\beta_1+\dots+\beta_s)} > p^{\beta_1+\dots+\beta_s} = p^e$$

where the inequality follows from the fact that we must have $s > 1$. □

3. A conjecture

Although the example in section 2 settles the question raised in [LS], one might still raise the question of whether such examples exist over "nice" rings, e.g., normal domains.

Let $\mathbb{K}$ be a field of prime characteristic p, let $R = \mathbb{K}[\![x, y, z, u, v, w]\!]$ and let I be the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$.

The ring $S = R/I$ is a normal, Cohen-Macaulay domain (cf. Theorem 7.3.1 in [BH].) Let E_S be the injective hull of the residue field of S and, as before, for all $e \geq 1$ let $\mathcal{F}_{<e}$ be the S-subalgebra of $\mathcal{F}^e(E_S)$ generated by $\mathcal{F}^1(E_S), \dots, \mathcal{F}^{e-1}(E_S)$. Note that $\mathcal{F}^0(E_S) = S$.

Conjecture 3.1. *For all $e \geq 1$, $\mathcal{F}^e(E_S)$ is not contained in $\mathcal{F}_{<e}$ and hence $\mathcal{F}^e(E_S)$ is not a finitely generated S-algebra.*

I have tested this conjecture using the computer system Macaulay2 ([GS]), and, for example, in characteristic 2, it holds for $1 \leq e \leq 6$.

References

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